

## GROWTH OF ÉTALE GROUPOIDS AND SIMPLE ALGEBRAS

VOLODYMYR NEKRASHEVYCH

ABSTRACT. We study growth and complexity of étale groupoids in relation to growth of their convolution algebras. As an application, we construct simple finitely generated algebras of arbitrary Gelfand-Kirillov dimension  $\geq 2$  and simple finitely generated algebras of quadratic growth over arbitrary fields.

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## 1. INTRODUCTION

Topological groupoids are extensively used in dynamics, topology, non-commutative geometry, and  $C^*$ -algebras, see [13, 29, 30]. With recent results on topological full groups (see [22, 15, 16]) new applications of groupoids to group theory were discovered.

Our paper studies growth and complexity for étale groupoids with applications to the theory of growth and Gelfand-Kirillov dimension of algebras. We give examples of groupoids whose convolution algebras (over an arbitrary field) have prescribed growth. In particular, we give first examples of simple algebras of quadratic growth over finite fields and simple algebras of Gelfand-Kirillov dimension 2 that do not have quadratic growth.

A *groupoid*  $\mathfrak{G}$  is the set of isomorphisms of a small category, i.e., a set  $\mathfrak{G}$  with partially defined multiplication and everywhere defined operation of taking inverse satisfying the following axioms:

- (1) If the products  $ab$  and  $bc$  are defined, then  $(ab)c$  and  $a(bc)$  are defined and are equal.

- (2) The products  $a^{-1}a$  and  $bb^{-1}$  are always defined and satisfy  $abb^{-1} = a$  and  $a^{-1}ab = b$  whenever the product  $ab$  is defined.

It follows from the axioms that  $(a^{-1})^{-1} = a$  and that a product  $ab$  is defined if and only if  $bb^{-1} = a^{-1}a$ . The elements of the form  $aa^{-1}$  are called *units* of the groupoid. We call  $\mathfrak{o}(g) = g^{-1}g$  and  $\mathfrak{t}(g) = gg^{-1}$  the *origin* and the *target* of the element  $g \in \mathfrak{G}$ .

A *topological groupoid* is a groupoid together with topology such that multiplication and taking inverse are continuous. It is called *étale* if every element has a basis of neighborhoods consisting of *bisections*, i.e., sets  $F$  such that  $\mathfrak{o} : F \rightarrow \mathfrak{o}(F)$  and  $\mathfrak{t} : F \rightarrow \mathfrak{t}(F)$  are homeomorphisms.

For example, if  $G$  is a discrete group acting (from the left) by homeomorphisms on a topological space  $\mathcal{X}$ , then the topological space  $G \times \mathcal{X}$  has a natural structure of an étale groupoid with respect to the multiplication

$$(g_1, g_2(x)) \cdot (g_2, x) = (g_1g_2, x).$$

In some sense étale groupoids are generalization of actions of discrete groups on topological spaces.

We consider two growth functions for an étale groupoid  $\mathfrak{G}$  with compact totally disconnected space of units. The first one is the most straightforward and classical: growth of fibers of the origin map. If  $S$  is an open compact generating set of  $\mathfrak{G}$  then, for a given unit  $x$ , we can consider the growth function  $\gamma_S(r, x)$  equal to the number of groupoid elements with origin in  $x$  that can be expressed as a product of at most  $n$  elements of  $S \cup S^{-1}$ . This notion of growth of a groupoid has appeared in many situations, especially in amenability theory for topological groupoids, see [17, 1]. See also Theorem 3.1 of our paper, where for a class of groupoids we show how sub-exponential growth implies absence of free subgroups in the topological full group of the groupoid.

This notion of growth does not capture full complexity of a groupoid precisely because it is “fiberwise”. Therefore, we introduce the second growth function: complexity of the groupoid. Let  $\mathcal{S}$  be a finite covering by open bisections of an open compact generating set  $S$  of  $\mathfrak{G}$ . For a given natural number  $r$  and units  $x, y \in \mathfrak{G}^{(0)}$  we write  $x \sim_r y$  if for any two products  $S_1S_2 \dots S_n$  and  $R_1R_2 \dots R_m$  of elements of  $\mathcal{S} \cup \mathcal{S}^{-1}$  such that  $n, m \leq r$  we have  $S_1S_2 \dots S_nx = R_1R_2 \dots R_my$  if and only if  $S_1S_2 \dots S_ny = R_1R_2 \dots R_mx$ . In other words,  $x \sim_r y$  if and only if balls of radius  $r$  with centers in  $x$  and  $y$  in the natural  $\mathcal{S}$ -labeled *Cayley graphs* of  $\mathfrak{G}$  are isomorphic. Then the *complexity function*  $\delta(r, \mathcal{S})$  is the number of  $\sim_r$ -equivalence classes of points of  $\mathfrak{G}^{(0)}$ .

This notion of complexity (called in this case *factor complexity*, or *subword complexity*) is well known and studied for groupoids of the action of shifts on closed shift-invariant subsets of  $X^{\mathbb{Z}}$ , where  $X$  is a finite alphabet. There is an extensive literature on it, see [8, 10]. An interesting result from the group-theoretic point of view is a theorem of N. Matte Bon [21] stating that if complexity of a subshift is strictly sub-quadratic, then the topological full group of the corresponding groupoid is Liouville. Here the *topological full group* of an étale groupoid  $\mathfrak{G}$  is the group of all  $\mathfrak{G}$ -bisections  $A$  such that  $\mathfrak{o}(A) = \mathfrak{t}(A) = \mathfrak{G}^{(0)}$ .

It seems that complexity of groupoids in more general étale groupoids has not been well studied yet. It would be interesting to understand how complexity function (together with the growth of fibers) is related with the properties of the topological full group of an étale groupoid. For example, it would be interesting to know if there exists a non-amenable (e.g., free) group acting faithfully on a compact topological space so that the corresponding groupoid of germs has sub-exponential growth and sub-exponential complexity functions.

We relate growth and complexity of groupoids with growth of algebras naturally associated with them. Suppose that  $\mathcal{A}$  is a finitely generated algebra with a unit over a field  $\mathbb{k}$ . Let  $V$  be the  $\mathbb{k}$ -linear span of a finite generating set containing the unit. Denote by  $V^n$  the linear span of all products  $a_1 a_2 \dots a_n$  for  $a_i \in V$ . Then  $\mathcal{A} = \bigcup_{n=1}^{\infty} V^n$ . *Growth* of  $\mathcal{A}$  is the function

$$\gamma(n) = \dim V^n.$$

It is easy to see that if  $\gamma_1, \gamma_2$  are growth functions defined using different finite generating sets, then there exists  $C > 1$  such that  $\gamma_1(n) \leq \gamma_2(Cn)$  and  $\gamma_2(n) \leq \gamma_1(Cn)$ .

*Gelfand-Kirillov* dimension of  $\mathcal{A}$  is defined as  $\limsup_{n \rightarrow \infty} \frac{\log \dim V^n}{\log n}$ , which informally is the degree of polynomial growth of the algebra. If  $\mathcal{A}$  is not finitely generated, then its Gelfand-Kirillov dimension is defined as the supremum of the Gelfand-Kirillov dimensions of all its sub-algebras. See the monograph [19] for a survey of results on growth of algebras and their Gelfand-Kirillov dimension.

It is known, see [34] and [19, Theorem 2.9], that Gelfand-Kirillov dimension can be any number in the set  $\{0, 1\} \cup [2, \infty]$ . The values in the interval  $(1, 2)$  are prohibited by a theorem of G.M. Bergman, see [19, Theorem 2.5]. There are examples of prime algebras of arbitrary Gelfand-Kirillov dimension  $d \in [2, \infty]$ , see [33], but it seems that no examples of simple algebras of arbitrary Gelfand-Kirillov dimension over finite fields were known so far.

A naturally defined *convolution algebra*  $\mathbb{k}[\mathfrak{G}]$  over arbitrary field  $\mathbb{k}$  is associated with every étale groupoid  $\mathfrak{G}$  with totally disconnected space of units. If the groupoid  $\mathfrak{G}$  is Hausdorff, then  $\mathbb{k}[\mathfrak{G}]$  is the convolution algebra of all continuous functions  $f : \mathfrak{G} \rightarrow \mathbb{k}$  with compact support, where  $\mathbb{k}$  is taken with the discrete topology. Here convolution  $f_1 \cdot f_2$  of two functions is the function given by the formula

$$f(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

In the non-Hausdorff case we follow A. Connes [9] and B. Steinberg [32], and define  $\mathbb{k}[\mathfrak{G}]$  as the linear span of the functions that are continuous on open compact subsets of  $\mathfrak{G}$ . Equivalently,  $\mathbb{k}[\mathfrak{G}]$  is the linear span of the characteristic functions of open compact  $\mathfrak{G}$ -bisections.

Note that the set  $\mathcal{B}(\mathfrak{G})$  of all open compact  $\mathfrak{G}$ -bisections (together with the empty one) is a semigroup. The algebra  $\mathbb{k}[\mathfrak{G}]$  is isomorphic to the quotient of the semigroup algebra of  $\mathcal{B}(\mathfrak{G})$  by the ideal generated by the relations  $F - (F_1 + F_2)$  for all triples  $F, F_1, F_2 \in \mathcal{B}(\mathfrak{G})$  such that  $F = F_1 \cup F_2$  and  $F_1 \cap F_2 = \emptyset$ .

We prove the following relation between growth of groupoids and growth of their convolution algebras.

**Theorem 1.1.** *Let  $\mathfrak{G}$  be an étale groupoid with compact totally disconnected space of units. Let  $\mathcal{S}$  be a finite set of open compact  $\mathfrak{G}$ -bisections such that  $S = \bigcup \mathcal{S}$  is a*

generating set of  $\mathfrak{G}$ . Let  $V \subset \mathbb{k}[\mathfrak{G}]$  be the linear span of the characteristic functions of elements of  $\mathcal{S}$ . Then

$$\dim V^n \leq \overline{\gamma}(r, \mathcal{S}) \delta(r, \mathcal{S}),$$

where  $\overline{\gamma}(r, \mathcal{S}) = \max_{x \in \mathfrak{G}^{(0)}} \gamma_{\mathcal{S}}(r, x)$ .

We say that a groupoid  $\mathfrak{G}$  is *minimal* if every  $\mathfrak{G}$ -orbit is dense in  $\mathfrak{G}^{(0)}$ . We say that  $\mathfrak{G}$  is *essentially principal* if the set of points  $x$  with trivial isotropy group is dense in  $\mathfrak{G}^{(0)}$ . Here the isotropy group of a point  $x$  is the set  $\{g \in \mathfrak{G} : \mathfrak{o}(g) = \mathfrak{t}(g) = x\}$ . It is known, see [7], that for a Hausdorff minimal essentially principal groupoid  $\mathfrak{G}$  with compact totally disconnected set of units the algebra  $\mathbb{k}[\mathfrak{G}]$  is simple. We give a proof of this fact for completeness in Proposition 4.1.

We give in Proposition 4.4 a condition (related to the classical notion of an *expansive dynamical system*) ensuring that  $\mathbb{k}[\mathfrak{G}]$  is finitely generated.

Fibers of the origin map provide us with naturally defined  $\mathbb{k}[\mathfrak{G}]$ -modules. Namely, for a given unit  $x \in \mathfrak{G}^{(0)}$  consider the vector space  $\mathbb{k}\mathfrak{G}_x$  of functions  $\phi : \mathfrak{G}_x \rightarrow \mathbb{k}$  with finite support, where  $\mathfrak{G}_x = \mathfrak{o}^{-1}(x)$  is the set of elements of the groupoid  $\mathfrak{G}$  with origin in  $x$ . Then convolution  $f \cdot \phi$  for any  $f \in \mathbb{k}[\mathfrak{G}]$  and  $\phi \in \mathbb{k}\mathfrak{G}_x$  is an element of  $\mathbb{k}\mathfrak{G}_x$ , and hence  $\mathbb{k}\mathfrak{G}_x$  is a left  $\mathbb{k}[\mathfrak{G}]$ -module.

It is easy to prove that if the isotropy group of  $x$  is trivial, then  $\mathbb{k}\mathfrak{G}_x$  is simple and that growth of  $\mathbb{k}\mathfrak{G}_x$  is bounded by  $\gamma_{\mathcal{S}}(x, r)$ , see Proposition 4.8.

As an example of applications of these results, we consider the following family of algebras. Let  $X$  be a finite alphabet, and let  $w : \mathbb{Z} \rightarrow X$  be a bi-infinite sequence of elements of  $X$ . Denote by  $D_x$ , for  $x \in X$  the diagonal matrix  $(a_{i,j})_{i,j \in \mathbb{Z}}$  given by

$$a_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ and } w(i) = x, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $T$  be the matrix  $(t_{i,j})_{i,j \in \mathbb{Z}}$  of the shift given by

$$t_{i,j} = \begin{cases} 1 & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Fix a field  $\mathbb{k}$ , and let  $\mathcal{A}_w$  be the  $\mathbb{k}$ -algebra generated by the matrices  $D_x$ , for  $x \in X$ , by  $T$ , and its transpose  $T^\top$ .

We say that  $w$  is *minimal* if for every finite subword  $(w(n), w(n+1), \dots, w(n+k))$  there exists  $R > 0$  such that for any  $i \in \mathbb{Z}$  there exists  $j \in \mathbb{Z}$  such that  $|i - j| \leq R$  and  $(w(j), w(j+1), \dots, w(j+k)) = (w(n), w(n+1), \dots, w(n+k))$ . We say that  $w$  is *non-periodic* if there does not exist  $p \neq 0$  such that  $w(n+p) = w(n)$  for all  $n \in \mathbb{Z}$ . *Complexity function*  $p_w(n)$  of the sequence  $w \in X^{\mathbb{Z}}$  is the number of different subwords  $(w(i), w(i+1), \dots, w(i+n-1))$  of length  $n$  in  $w$ .

The following theorem is a corollary of the results of our paper, see Subsection 4.4.1 and Example 4.6.

**Theorem 1.2.** *Suppose that  $w \in X^{\mathbb{Z}}$  is minimal and non-periodic. Then the algebra  $\mathcal{A}_w$  is simple, and its growth  $\gamma(n)$  satisfies*

$$C^{-1}n \cdot p_w(C^{-1}n) \leq Cn \cdot p_w(Cn)$$

for some  $C > 1$ .

We can apply now results on complexity of sequences to construct simple algebras of various growths. For example, if  $w$  is *Sturmian*, then  $p_w(n) = n + 1$ , and hence  $\mathcal{A}_w$  has quadratic growth. For different *Toeplitz* sequences we can obtain simple

algebras of arbitrary Gelfand-Kirillov dimension  $d \geq 2$ , or simple algebras of growth  $n \log n$ , etc., see Subsection 4.4.1.

Another class of examples of groupoids considered in our paper are groupoids associated with groups acting on a rooted tree. If  $G$  acts by automorphisms on a locally finite rooted tree  $T$ , then it acts by homeomorphisms on the boundary  $\partial T$ . One can consider the *groupoid of germs*  $\mathfrak{G}$  of the action. Convolution algebras  $\mathbb{k}[\mathfrak{G}]$  are related to the *thinned algebras* studied in [31, 2]. In the case when  $G$  is a *contracting self-similar group*, Theorem 1.1 implies a result of L. Bartholdi from [2] giving an estimate of Gelfand-Kirillov dimension for the thinned algebras of contracting self-similar groups.

## 2. ÉTALE GROUPOIDS

A *groupoid* is a small category of isomorphisms (more precisely, the set of its morphisms). For a groupoid  $\mathfrak{G}$ , we denote by  $\mathfrak{G}^{(2)}$  the set of composable pairs, i.e., the set of pairs  $(g_1, g_2) \in \mathfrak{G} \times \mathfrak{G}$  such that the product  $g_1 g_2$  is defined. We denote by  $\mathfrak{G}^{(0)}$  the set of units of  $\mathfrak{G}$ , i.e., the set of identical isomorphisms. We also denote by  $\circ, \mathfrak{t} : \mathfrak{G} \rightarrow \mathfrak{G}^{(0)}$  the *origin* and *target* maps given by

$$\circ(g) = g^{-1}g, \quad \mathfrak{t}(g) = gg^{-1}.$$

We interpret then an element  $g \in \mathfrak{G}$  as an arrow from  $\circ(g)$  to  $\mathfrak{t}(g)$ . The product  $g_1 g_2$  is defined if and only if  $\mathfrak{t}(g_2) = \circ(g_1)$ .

For  $x \in \mathfrak{G}^{(0)}$ , denote

$$\mathfrak{G}_x = \{g \in \mathfrak{G} : \circ(g) = x\}, \quad \mathfrak{G}^x = \{g \in \mathfrak{G} : \mathfrak{t}(g) = x\}.$$

The set  $\mathfrak{G}_x \cap \mathfrak{G}^x$  is called the *isotropy group* of  $x$ . A groupoid is said to be *principal* (or an equivalence relation) if the isotropy group of every point is trivial. Two units  $x, y \in \mathfrak{G}^{(0)}$  belong to one *orbit* if there exists  $g \in \mathfrak{G}$  such that  $\circ(g) = x$  and  $\mathfrak{t}(g) = y$ . It is easy to see that belonging to one orbit is an equivalence relation.

A *topological groupoid* is a groupoid  $\mathfrak{G}$  with a topology on it such that multiplication  $\mathfrak{G}^{(2)} \rightarrow \mathfrak{G}$  and taking inverse  $\mathfrak{G} \rightarrow \mathfrak{G}$  are continuous maps. We do not require that  $\mathfrak{G}$  is Hausdorff, though we assume that the space of units  $\mathfrak{G}^{(0)}$  is metrizable and locally compact.

A  $\mathfrak{G}$ -*bisection* is a subset  $F \subset \mathfrak{G}$  such that the maps  $\circ : F \rightarrow \circ(F)$  and  $\mathfrak{t} : F \rightarrow \mathfrak{t}(F)$  are homeomorphisms.

**Definition 2.1.** A topological groupoid  $\mathfrak{G}$  is *étale* if the set of all open  $\mathfrak{G}$ -bisections is a basis of the topology of  $\mathfrak{G}$ .

Let  $\mathfrak{G}$  be an étale groupoid. It is easy to see that product of two open bisections is an open bisection. It follows that for every bisection  $F$  the sets  $\circ(F) = F^{-1}F$  and  $\mathfrak{t}(F) = FF^{-1}$  are open, which in turn implies that  $\mathfrak{G}^{(0)}$  is an open subset of  $\mathfrak{G}$ .

If  $\mathfrak{G}$  is not Hausdorff, then there exist  $g_1, g_2 \in \mathfrak{G}$  that do not have disjoint bisections. Since  $\mathfrak{G}^{(0)}$  is Hausdorff, this implies that  $\circ(g_1) = \circ(g_2)$  and  $\mathfrak{t}(g_1) = \mathfrak{t}(g_2)$ . It follows that the unit  $x = \circ(g_1)$  and the element  $g_2^{-1}g_1$  of the isotropy group of  $x$  do not have disjoint open neighborhoods. In particular, it means that principal étale groupoids are always Hausdorff, and that an étale groupoid is Hausdorff if and only if  $\mathfrak{G}^{(0)}$  is a closed subset of  $\mathfrak{G}$ .

**Example 2.1.** Let  $G$  be a discrete group acting by homeomorphisms on a space  $\mathcal{X}$ . Then the space  $G \times \mathcal{X}$  has a natural groupoid structure with given by the

multiplication

$$(g_2, g_1(x))(g_1, x) = (g_2 g_1, x).$$

This is an étale groupoid, since every set  $\{g\} \times \mathcal{X}$  is an open bisection. The groupoid  $G \times \mathcal{X}$  is called the *groupoid of the action*, and is denoted  $G \ltimes \mathcal{X}$ .

Our main class of groupoids will be naturally defined quotients of the groupoids of actions, called groupoids of germs.

**Example 2.2.** Let  $G$  and  $\mathcal{X}$  be as in the previous example. A *germ* is an equivalence class of a pair  $(g, x) \in G \times \mathcal{X}$  where  $(g_1, x)$  and  $(g_2, x)$  are equivalent if there exists a neighborhood  $U$  of  $x$  such that the maps  $g_1 : U \rightarrow \mathcal{X}$  and  $g_2 : U \rightarrow \mathcal{X}$  coincide. The set of germs is also an étale groupoid with the same multiplication rule as in the previous example. We call it *groupoid of germs of the action*.

The spaces of units in both groupoids are naturally identified with the space  $\mathcal{X}$  (namely, we identify the pair or the germ  $(1, x)$  with  $x$ ). The groupoid of the action is Hausdorff if  $\mathcal{X}$  is Hausdorff, since it is homeomorphic to  $G \times \mathcal{X}$ . The groupoid of germs, on the other hand, is frequently non-Hausdorff, even for a Hausdorff space  $\mathcal{X}$ .

If every germ of every non-trivial element of  $G$  is not a unit (i.e., not equal to a germ of the identical homeomorphism), then the groupoid of the action coincides with the groupoid of germs.

Many interesting examples of étale groupoids appear in dynamics and topology, see [13, 6, 27].

### 3. COMPACTLY GENERATED GROUPOIDS

For the rest of the paper,  $\mathfrak{G}$  is an étale groupoid such that  $\mathfrak{G}^{(0)}$  is a compact totally disconnected metrizable space. Note that then there exists a basis of topology of  $\mathfrak{G}$  consisting of open compact  $\mathfrak{G}$ -bisections. Note that we allow compact non-closed and compact non-Hausdorff sets, since  $\mathfrak{G}$  in general is not Hausdorff. However, if  $F$  is an open compact bisection, then  $\mathfrak{o}(F)$  and  $\mathfrak{t}(F)$  are clopen (i.e., closed and open) and  $F$  is Hausdorff.

#### 3.1. Cayley graphs and their growth.

**Definition 3.1.** A groupoid  $\mathfrak{G}$  with compact totally disconnected unit space is *compactly generated* if there exists a open compact subset  $S \subset \mathfrak{G}$  such that  $\mathfrak{G} = \bigcup_{n \geq 0} (S \cup S^{-1})^n$ . The set  $S$  is called the *generating set* of  $\mathfrak{G}$ .

This definition is equivalent (for étale groupoids with compact totally disconnected unit space) to the definition of [14].

**Example 3.1.** Let  $G$  be a group acting on a Cantor set  $\mathcal{X}$ . If  $S$  is a finite generating set of  $G$ , then  $S \times \mathcal{X}$  is an open compact generating set of the groupoid  $G \ltimes \mathcal{X}$ . The set all of germs of elements of  $S$  is an open compact generating set of the groupoid of germs of the action. Thus, both groupoids are compactly generated if  $G$  is finitely generated.

Let  $S$  be an open compact generating set of  $\mathfrak{G}$ . Let  $x \in \mathfrak{G}^{(0)}$ . The *Cayley graph*  $\mathfrak{G}(x, S)$  is the directed graph with the set of vertices  $\mathfrak{G}_x$  in which we have an arrow from  $g_1$  to  $g_2$  whenever there exists  $s \in S$  such that  $g_2 = s g_1$ .

We will often consider the graph  $\mathfrak{G}(x, S)$  as a *rooted graph* with root  $x$ . Morphism  $\phi : \Gamma_1 \longrightarrow \Gamma_2$  of rooted graphs is a morphism of graphs that maps the root of  $\Gamma_1$  to the root of  $\Gamma_2$ .

Note that since  $S$  can be covered by a finite set of bisections, the degrees of vertices of the graphs  $\mathfrak{G}(x, S)$  are uniformly bounded.

**Example 3.2.** Let  $G$  be a finitely generated group acting on a totally disconnected compact space  $\mathcal{X}$ . Let  $S$  be a finite generating set of  $G$ , and let  $S \times \mathcal{X}$  be the corresponding generating set of the groupoid of action  $G \ltimes \mathcal{X}$ . The Cayley graphs  $G \ltimes \mathcal{X}(x, S \times \mathcal{X})$  coincide then with the Cayley graphs of  $G$  (with respect to the generating set  $S$ ).

The groupoid of germs  $\mathfrak{G}$  will have smaller Cayley graphs. Let  $S' \subset \mathfrak{G}$  be the set of all germs of elements of  $S$ . Denote, for  $x \in \mathcal{X}$ , by  $G_{(x)}$  the subgroup of  $G$  consisting of all elements  $g \in G$  such that there exists a neighborhood  $U$  of  $x$  such that  $g$  fixes every point of  $U$ . Then  $\mathfrak{G}(x, S')$  is isomorphic to the *Schreier graph* of  $G$  modulo  $G_{(x)}$ . Its vertices are the cosets  $hG_{(x)}$ , and a coset  $h_1G_{(x)}$  is connected by an arrow with  $h_2G_{(x)}$  if there exists a generator  $s \in S$  such that  $sh_1G_{(x)} = h_2G_{(x)}$ .

Cayley graphs  $\mathfrak{G}(x, S)$  are closely related to the *orbital graphs*, which are defined as graphs  $\Gamma(x, S)$  with the set of vertices equal to the orbit of  $x$ , in which a vertex  $x_1$  is connected by an arrow to a vertex  $x_2$  if there exists  $g \in S$  such that  $o(s) = x_1$  and  $t(s) = x_2$ . Orbital graph  $\Gamma(x, S)$  is the quotient on the Cayley graph  $\mathfrak{G}(x, S)$  by the natural right action of the isotropy group of  $x$ . In particular, orbital graph and the Cayley graph coincide if the isotropy group of  $x$  is trivial.

Denote by  $B_S(x, n)$  the ball of radius  $n$  with center  $x$  in the graph  $\mathfrak{G}(x, S)$  seen as a rooted graph (with root  $x$ ). Let

$$\gamma_S(x, n) = |B_S(x, n)|, \quad \bar{\gamma}(n, S) = \max_{x \in \mathfrak{G}^{(0)}} \gamma_S(x, n).$$

If  $S_1$  and  $S_2$  are two open compact generating sets of  $\mathfrak{G}$ , then there exists  $m$  such that  $S_2 \subset \bigcup_{1 \leq k \leq m} (S_1 \cup S_1^{-1})^k$  and  $S_1 \subset \bigcup_{1 \leq k \leq m} (S_2 \cup S_2^{-1})^k$ . Then  $\gamma_{S_1}(x, mn) \geq \gamma_{S_2}(x, n)$  and  $\gamma_{S_2}(x, mn) \geq \gamma_{S_1}(x, n)$  for all  $n$ . It also follows that  $\bar{\gamma}(mn, S_1) \geq \bar{\gamma}(n, S_2)$  and  $\bar{\gamma}(mn, S_2) \geq \bar{\gamma}(n, S_1)$  for all  $n$ . In other words, the *growth rate* of the functions  $\gamma_S(x, n)$  and  $\bar{\gamma}(n, S)$  do not depend on the choice of  $S$ , if  $S$  is a generating set.

Condition of polynomial growth of Cayley graphs of groupoids (or, in the measure-theoretic category, of connected components of graphings of equivalence relations) appear in the study of amenability of groupoids, see [17, 1].

Here is another example of applications of the notion of growth of groupoids.

**Theorem 3.1.** *Let  $G$  be a finitely generated subgroup of the automorphism group of a locally finite rooted tree  $T$ . Consider the groupoid of germs  $\mathfrak{G}$  of the action of  $G$  on the boundary  $\partial T$  of the tree. If  $\gamma_S(x, n)$  has sub-exponential growth for every  $x \in \partial T$ , then  $G$  has no free subgroups.*

*Proof.* By [26, Theorem 3.3], if  $G$  has a free subgroup, then either there exists a free subgroup  $F$  and a point  $x \in \partial T$  such that the stabilizer of  $x$  in  $F$  is trivial, or there exists a free subgroup  $F$  and a point  $x \in \partial T$  such that  $x$  is fixed by  $F$  and every non-trivial element  $g$  of  $F$  the germ  $(g, x)$  is non-trivial. But both conditions imply that the Cayley graph  $\mathfrak{G}(x, S)$  has exponential growth.  $\square$

**3.2. Complexity.** Let  $\mathcal{S}$  be a finite set of open compact  $\mathfrak{G}$ -bisections such that  $S = \bigcup \mathcal{S}$  is a generating set. Note that every compact subset of  $\mathfrak{G}$  can be covered by a finite number of open compact  $\mathfrak{G}$ -bisections.

Denote by  $\mathfrak{G}(x, \mathcal{S})$  the oriented labeled graph with the set of vertices  $\mathfrak{G}_x$  in which we have an arrow from  $g_1$  to  $g_2$  labeled by  $A \in \mathcal{S}$  if there exists  $s \in A$  such that  $g_2 = sg_1$ .

The graph  $\mathfrak{G}(x, \mathcal{S})$  basically coincides with  $\mathfrak{G}(x, S)$  for  $S = \bigcup \mathcal{S}$ . The only difference is the labeling and that some arrows of  $\mathfrak{G}(x, S)$  become multiple arrows in  $\mathfrak{G}(x, \mathcal{S})$ . In particular, the metrics induced on the sets of vertices of graphs  $\mathfrak{G}(x, S)$  and  $\mathfrak{G}(x, \mathcal{S})$  coincide.

We denote by  $B_{\mathcal{S}}(x, r)$  or just by  $B(x, r)$  the ball of radius  $r$  with center in  $x$ , seen as a rooted oriented labeled graph. We write  $x \sim_r y$  if  $B_{\mathcal{S}}(x, r)$  and  $B_{\mathcal{S}}(y, r)$  are isomorphic.

**Definition 3.2.** *Complexity* of  $\mathcal{S}$  is the function  $\delta(r, \mathcal{S})$  equal to the number of  $\sim_r$ -equivalence classes.

It is easy to see that  $\delta(r, \mathcal{S})$  is finite for every  $r$  and  $\mathcal{S}$ .

### 3.3. Examples.

**3.3.1. Shifts.** Let  $X$  be a finite alphabet containing more than one letter. Consider the space  $X^{\mathbb{Z}}$  of all bi-infinite words over  $X$ , i.e., maps  $w : \mathbb{Z} \rightarrow X$ . Denote by  $s : X^{\mathbb{Z}} \rightarrow X^{\mathbb{Z}}$  the shift map given by the rule  $s(w)(i) = w(i + 1)$ . The space  $X^{\mathbb{Z}}$  is homeomorphic to the Cantor set with respect to the direct product topology (where  $X$  is discrete).

A *sub-shift* is a closed  $s$ -invariant subset  $\mathcal{X} \subset X^{\mathbb{Z}}$ . We always assume that  $\mathcal{X}$  has no isolated points. For a sub-shift  $\mathcal{X}$ , consider the groupoid  $\mathfrak{G}$  of the germs of the action of  $\mathbb{Z}$  on  $\mathcal{X}$  generated by the shift. It is easy to see that all germs of non-zero powers of the shift are non-trivial, hence the groupoid  $\mathfrak{G}$  coincides with the groupoid  $\mathbb{Z} \ltimes \mathcal{X}$  of the action. As usual, we will identify  $\mathcal{X}$  with the space of units  $\mathfrak{G}^{(0)}$ . The set  $S = \{(s, x) : x \in \mathcal{X}\}$  is an open compact generating set of  $\mathfrak{G}$ . The Cayley graphs  $\mathfrak{G}(w, S)$  are isomorphic to the Cayley graph of  $\mathbb{Z}$  with respect to the generating set  $\{1\}$ .

If  $\mathcal{X}$  is *aperiodic*, i.e., if it does not contain periodic sequences, then  $\mathfrak{G}$  is principal. Note that  $\mathfrak{G}$  is always Hausdorff.

For  $x \in X$ , denote by  $S_x$  set of germs of the restriction of  $s$  onto the cylindrical set  $\{w \in \mathcal{X} : w(0) = x\}$ . Then  $\mathcal{S} = \{S_x\}_{x \in X}$  is a covering of  $S$  by disjoint clopen subsets of  $S$ . Then for every  $w \in \mathcal{X}$ , the Cayley graph  $\mathfrak{G}(w, \mathcal{S})$  basically repeats  $w$ : its set of vertices is the set of germs  $(s^n, w)$ ,  $n \in \mathbb{Z}$ ; for every  $n$  we have an arrow from  $(s^n, w)$  to  $(s^{n+1}, w)$  labeled by  $S_{w(n)}$ .

In particular, we have

$$\delta(n, \mathcal{S}) = p_{\mathcal{X}}(2n),$$

where  $p_{\mathcal{X}}(k)$  denotes the number of words of length  $k$  that appear as subwords of elements of  $\mathcal{X}$ .

Complexity  $p_{\mathcal{X}}(n)$  of subshifts is a well studied subject, see [20, 10, 8] and references therein.

Two classes of subshifts are especially interesting for us: Sturmian and Toeplitz subshifts.



Let  $\theta \in (0, 1)$  be an irrational number, and consider the rotation

$$R_\theta : x \mapsto x + \theta \pmod{1}$$

of the circle  $\mathbb{R}/\mathbb{Z}$ . For a number  $x \in \mathbb{R}/\mathbb{Z}$  not belonging to the  $R_\theta$ -orbit of 0, consider the  $\theta$ -itinerary  $I_{\theta,x} \in \{0, 1\}^{\mathbb{Z}}$  given by

$$I_{\theta,x}(n) = \begin{cases} 0 & \text{if } x + n\theta \in (0, \theta) \pmod{1}, \\ 1 & \text{if } x + n\theta \in (\theta, 1) \pmod{1}. \end{cases}$$

In other words,  $I_{\theta,x}$  describes the itinerary of  $x \in \mathbb{R}/\mathbb{Z}$  under the rotation  $R_\theta$  with respect to the partition  $[0, \theta), [\theta, 1)$  of the circle  $\mathbb{R}/\mathbb{Z}$ . If  $x$  belongs to the orbit of 0, then we define two itineraries  $I_{\theta,x+0} = \lim_{t \rightarrow x+0} I_{\theta,t}$  and  $I_{\theta,x-0} = \lim_{t \rightarrow x-0} I_{\theta,t}$ , where  $t$  in the limits belongs to the complement of the orbit of 0.

The set  $\mathcal{X}_\theta$  of all itineraries is a subshift of  $\{0, 1\}^{\mathbb{Z}}$  called the *Sturmian subshift* associated with  $\theta$ . Informally, the space  $\mathcal{X}_\theta$  is obtained from the circle  $\mathbb{R}/\mathbb{Z}$  by “cutting” it along the  $R_\theta$ -orbit of 0, i.e., by replacing each point  $x = n\theta$  by two copies  $x + 0$  and  $x - 0$ . A basis of topology of  $\mathcal{X}_\theta$  is the set of arcs of the form  $[n\theta + 0, m\theta - 0]$ . The shift is identified in this model with the natural map induced by the rotation  $R_\theta$ .

Complexity  $p_{\mathcal{X}_\theta}(n)$  of the Sturmian subshift is equal to the number of all possible  $R_\theta$ -itineraries of length  $n$ . Consider the set  $\{R_\theta^{-k}(\theta)\}_{k=0,1,\dots,n}$ . It separates the circle  $\mathbb{R}/\mathbb{Z}$  into  $n + 1$  arcs such that two points  $x, y$  have equal length  $n$  segments  $\{0, \dots, n - 1\} \rightarrow \{0, 1\}$  of their itineraries  $I_{\theta,x}, I_{\theta,y}$  if and only if they belong to one arc. It follows that  $p_{\mathcal{X}_\theta}(n) = n + 1$ . The subshifts of the form  $\mathcal{X}_\theta$  and their elements are called *Sturmian subshifts* and *Sturmian sequences*.

A sequence  $w : \mathbb{Z} \rightarrow \mathbb{Z}$  is a *Toeplitz sequence* if it is not periodic and for every  $n \in \mathbb{Z}$  there exists  $p \in \mathbb{N}$  such that  $w(n + kp) = w(n)$  for all  $k \in \mathbb{Z}$ . Complexity of Toeplitz sequences is well studied.

It is known, for example, (see [20, Proposition 4.79]) that for any  $1 \leq \alpha \leq \beta \leq \infty$  there exists a Toeplitz subshift  $\mathcal{X}$  (i.e., closure of the shift orbit of a Toeplitz sequence) such that

$$\liminf_{n \rightarrow \infty} \frac{\ln p_{\mathcal{X}}(n)}{\ln n} = \alpha, \quad \limsup_{n \rightarrow \infty} \frac{\ln p_{\mathcal{X}}(n)}{\ln n} = \beta.$$

The following theorem is proved by M. Koskas in [18].

**Theorem 3.2.** *For every rational number  $p/q > 1$  and every positive increasing differentiable function  $f(x)$  satisfying  $f(n) = o(n^\alpha)$  for all  $\alpha > 0$ , and  $nf'(n) = o(n^\alpha)$  for all  $\alpha > 0$ , there exists a Toeplitz subshift  $\mathcal{X}$  and two constants  $c_1, c_2 > 0$  satisfying  $c_1 f(n) n^{p/q} \leq p_{\mathcal{X}}(n) \leq c_2 f(n) n^{p/q}$  for all  $n \in \mathbb{N}$ .*

**3.3.2. Groups acting on rooted trees.** Let  $X$  be a finite alphabet,  $|X| \geq 2$ . Denote by  $X^*$  the set of all finite words (including the empty word  $\emptyset$ ). We consider  $X^*$  as a rooted tree with root  $\emptyset$  in which every word  $v \in X^*$  is connected to the words of the form  $vx$  for all  $x \in X$ . The *boundary* of the tree is naturally identified with the space  $X^{\mathbb{N}}$  of all one-sided sequences  $x_1 x_2 x_3 \dots$ . Every automorphism of the rooted tree  $X^*$  naturally induces a homeomorphism of  $X^{\mathbb{N}}$ .

Let  $g$  be an automorphism of the tree  $X^*$ . For every  $v \in X^*$  there exists a unique automorphism  $g|_v$  of the tree  $X^*$  such that

$$g(vw) = g(v)g|_v(w)$$

for all  $w \in X^*$ . We say that a group  $G$  of automorphisms of  $X^*$  is *self-similar* if  $g|_v \in G$  for every  $g \in G$  and  $v \in X^*$ . For every  $v \in X^*$  and  $w \in X^{\mathbb{N}}$  the germ  $(g, vw)$  depends only on the quadruple  $(v, g(v), g|_v, w)$ .

**Example 3.3.** Consider the automorphism  $a$  of the binary tree  $\{0, 1\}^*$  defined by the recursive rules

$$a(0w) = 1w, \quad a(1w) = 0a(w).$$

It is called the *adding machine*, or *odometer*. The cyclic group generated by  $a$  is self-similar.

**Example 3.4.** Consider the automorphisms of  $\{0, 1\}^*$  defined by the recursive rules

$$a(0w) = 1w, a(1w) = 0w$$

and

$$\begin{aligned} b(0w) &= 0a(w), & b(1w) &= 1c(w), \\ c(0w) &= 0a(w), & c(1w) &= 1d(w), \\ d(0w) &= 0w, & d(1w) &= 1b(w). \end{aligned}$$

The group generated by  $a, b, c, d$  is the *Grigorchuk group*, see [12].

For more examples of self-similar groups and their applications, see [24].

Let  $G$  be a finitely generated self-similar group, and let  $l(g)$  denote the length of an element  $g \in G$  with respect to some fixed finite generating set of  $G$ . The *contraction coefficient* of the group  $G$  is the number

$$\lambda = \limsup_{n \rightarrow \infty} \limsup_{g \in G, l(g) \rightarrow \infty} \max_{v \in X^n} \frac{l(g|_v)}{l(g)}.$$

The group is said to be *contracting* if  $\lambda < 1$ .

For example, the adding machine action of  $\mathbb{Z}$  and the Grigorchuk group are both contracting with contraction coefficient  $\lambda = 1/2$ .

**Proposition 3.3.** *Let  $G$  be a contracting self-similar group acting on the tree  $X^*$ , and let  $\lambda$  be the contraction coefficient. Consider the groupoid of germs  $\mathfrak{G}$  of the action of  $G$  on  $X^{\mathbb{N}}$ , let  $S$  be a finite generating set of  $G$ , and let  $\mathcal{S}$  be the set of  $\mathfrak{G}$ -bisets of the form  $\{(s, w) : w \in X^{\mathbb{N}}\}$  for  $s \in S$ . Then we have*

$$\limsup_{n \rightarrow \infty} \frac{\log \overline{\gamma}(n, \mathcal{S})}{\log n} \leq \frac{\log |X|}{-\log \lambda}, \quad \limsup_{n \rightarrow \infty} \frac{\log \delta(n, \mathcal{S})}{\log n} \leq \frac{\log |X|}{-\log \lambda}.$$

*Proof.* Let  $\rho$  be any number in the interval  $(\lambda, 1)$ . Then there exist  $n_0, l_0$  such that for all elements  $g \in G$  such that  $l(g) > l_0$  we have  $l(g|_v) \leq \rho^{n_0} l(g)$  for all  $v \in X^{n_0}$ . It follows that there exists a finite set  $\mathcal{N}$  such that  $g|_v \in \mathcal{N}$  for all  $v \in X^*$  and for every  $g \in G \setminus \mathcal{N}$  we have  $l(g|_v) \leq \rho^{n_0} l(g)$  for all words  $v \in X^*$  of length at least  $n_0$ .

Then for every  $g \in G$  and for every word  $v \in X^*$  of length at least  $\left\lfloor \frac{\log l(g) - \log l_0}{-\log \rho} \right\rfloor + n_0$  we have  $g|_v \in \mathcal{N}$ . Let  $w = x_1 x_2 \dots \in X^{\mathbb{N}}$ , and denote  $v = x_1 x_2 \dots x_n$ ,  $w' = x_{n+1} x_{n+2} \dots$  for  $n = \left\lfloor \frac{\log r - \log l_0}{-\log \rho} \right\rfloor + n_0$ . Then for fixed  $w$  and all  $g$  such that  $l(g) \leq r$ , the germ  $(g, w)$  depends only on  $g(v)$  and  $g|_v$ . There are not more than  $|X|^n$  possibilities for  $g(v)$ , hence the number of germs  $(g, w)$  is not more than

$$|\mathcal{N}| \cdot |X|^n \leq |\mathcal{N}| \exp \left( \log |X| \left( \frac{\log r - \log l_0}{-\log \rho} + n_0 \right) \right) \leq C_1 r^{\frac{\log |X|}{-\log \rho}}$$

for  $C_1 = |\mathcal{N}| \cdot |X|^{\frac{\log l_0}{\log \rho} + n_0}$ . Consequently, for every  $\rho \in (\lambda, 1)$  there exists  $C_1 > 0$  such that

$$\overline{\gamma}(r, \mathcal{S}) \leq C_1 r^{\frac{\log |X|}{-\log \rho}},$$

hence  $\limsup_{r \rightarrow \infty} \frac{\log \overline{\gamma}(r, \mathcal{S})}{\log r} \leq \frac{\log |X|}{-\log \lambda}$ .

It is enough, in order to know the ball  $B_{\mathcal{S}}(w, r)$ , to know for every word  $g \in G$  of length at most  $2r$  whether the germ  $(g, w)$  is a unit. Let, as above,  $w = vw'$ , where length of  $v$  is  $n = \left\lfloor \frac{\log 2r - \log l_0}{-\log \rho} \right\rfloor + n_0$ . For every  $g \in G$  of length at most  $2r$  the germ  $(g, w)$  is a unit if and only if  $g(v) = v$  and  $(g|_v, w')$  is a unit. We have  $g|_v \in \mathcal{N}$ , so  $B_{\mathcal{S}}(w, r)$  depends only on  $v$  and the set  $T_{w'} = \{h \in \mathcal{N} : (h, w') \in \mathfrak{G}^{(0)}\}$ . Consequently,

$$\delta(r, \mathcal{S}) \leq 2^{|\mathcal{N}|} \cdot |X|^n \leq C_2 r^{\frac{\log |X|}{-\log \rho}},$$

where  $C_2 = 2^{|\mathcal{N}|} |X|^{\frac{\log l_0 - \log 2}{\log \rho} + n_0}$ , which shows that  $\limsup_{r \rightarrow \infty} \frac{\log \delta(r, \mathcal{S})}{\log r} \leq \frac{\log |X|}{-\log \lambda}$ .  $\square$

Both estimates in Proposition 3.3 are not sharp in general. For example, consider a self-similar action of  $\mathbb{Z}^2$  over the alphabet  $X$  of size 5 associated with the virtual endomorphism given by the matrix  $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}$ , see [24, 2.9, 2.12] and [28] for details. Note that the eigenvalues of  $A$  are  $\left(\frac{5 \pm \sqrt{5}}{2}\right)^{-1} \in (0, 1)$ , hence the contraction coefficient is  $\lambda = \frac{2}{5 - \sqrt{5}} = \frac{5 + \sqrt{5}}{10}$ . On the other hand  $\overline{\gamma}(r, \mathcal{S})$  grows as a quadratic polynomial, while  $\delta(r, \mathcal{S})$  is bounded.

#### 4. CONVOLUTION ALGEBRAS

**4.1. Definitions.** Let  $\mathfrak{G}$  be an étale groupoid, and let  $\mathbb{k}$  be a field. *Support* of a function  $f : \mathfrak{G} \rightarrow \mathbb{k}$  is closure of the set of points  $x \in \mathfrak{G}$  such that  $f(x) \neq 0$ . If  $f_1, f_2$  are functions with compact support, then their *convolution* is given by the formula

$$f_1 * f_2(g) = \sum_{h \in \mathfrak{G}_{o(g)}} f_1(gh^{-1})f_2(h).$$

Note that since  $f_2$  has compact support, the set of elements  $h \in \mathfrak{G}_{o(g)}$  such that  $f_2(h) \neq 0$  is finite.

It is easy to see that if  $f_1, f_2$  are supported on the space of units, then their convolution coincides with their pointwise product. If  $F_1, F_2$  are bisections, then their characteristic functions satisfy  $1_{F_1} * 1_{F_2} = 1_{F_1 F_2}$ .

The set of all functions  $f : \mathfrak{G} \rightarrow \mathbb{k}$  with compact support forms an algebra over  $\mathbb{k}$  with respect to convolution. But this algebra is too big, and its definition does not use the topology of  $\mathfrak{G}$  much. On the other hand, the algebra of all continuous functions (with discrete topology on  $\mathbb{k}$ ) is too small in the non-Hausdorff case. Therefore, we adopt the next definition, following Connes [9], see also [29] and [32].

**Definition 4.1.** The *convolution algebra*  $\mathbb{k}[\mathfrak{G}]$  is the  $\mathbb{k}$ -algebra generated by the characteristic functions  $1_F$  of open compact  $\mathfrak{G}$ -bisections (with respect to convolution).

If  $\mathfrak{G}$  is Hausdorff, then  $\mathbb{k}[\mathfrak{G}]$  is the algebra of all continuous (i.e., locally constant) functions  $f : \mathfrak{G} \rightarrow \mathbb{k}$ , where  $\mathbb{k}$  has discrete topology. In the non-Hausdorff case

the algebra  $\mathbb{k}[\mathfrak{G}]$  contains discontinuous functions (e.g., characteristic functions of non-closed open compact bisections).

From now on we will use the usual multiplication sign for convolution. The unit of the algebra  $\mathbb{k}[\mathfrak{G}]$  is the characteristic function of  $\mathfrak{G}^{(0)}$ , which we will often denote just by 1.

If  $\mathfrak{G} = G \ltimes \mathcal{X}$  is the groupoid of an action, then  $\mathbb{k}[\mathfrak{G}]$  is generated by the commutative algebra of locally constant functions  $f : \mathcal{X} \rightarrow \mathbb{k}$  (with pointwise multiplication and addition) and the group ring  $\mathbb{k}[G]$  subject to relations

$$g^{-1} \cdot f \cdot g = f \circ g,$$

for all  $f : \mathcal{X} \rightarrow \mathbb{k}$  and  $g \in G$ , where  $f \circ g : \mathcal{X} \rightarrow \mathbb{k}$  is given by  $(f \circ g)(x) = f(g(x))$ . In other words, it is the *cross-product* of the algebra of functions and the group ring.

Let  $\mathcal{T} \subset \mathfrak{G}^{(0)}$  be the set of units with trivial isotropy groups. The set  $\mathcal{T}$  is  $\mathfrak{G}$ -invariant, i.e., is a union of  $\mathfrak{G}$ -orbits.

**Definition 4.2.** We say that  $\mathfrak{G}$  is *essentially principal* if the set  $\mathcal{T}$  is dense in  $\mathfrak{G}^{(0)}$ . It is *principal* if  $\mathcal{T} = \mathfrak{G}^{(0)}$ . The groupoid  $\mathfrak{G}$  is said to be *minimal* if every  $\mathfrak{G}$ -orbit is dense in  $\mathfrak{G}^{(0)}$ .

**Example 4.1.** For every homeomorphism  $g$  of a metric space  $\mathcal{X}$ , the set of points  $x \in \mathcal{X}$  such that  $g(x) = x$  and the germ  $(g, x)$  is non-trivial is a closed nowhere dense set. It follows that if  $G$  is a countable group of homeomorphisms of  $\mathcal{X}$ , then groupoid of germs of the action is essentially principal.

Simplicity of essentially principal minimal groupoids is a well known fact, see [7] and a  $C^*$ -version in [30, Proposition 4.6]. We provide a proof of the following simple proposition just for completeness.

**Proposition 4.1.** *Suppose that  $\mathfrak{G}$  is essentially principal and minimal. Let  $I$  be the set of functions  $f \in \mathbb{k}[\mathfrak{G}]$  such that  $f(g) = 0$  for every  $g \in \mathfrak{G}$  such that  $\mathfrak{o}(g), \mathfrak{t}(g) \in \mathcal{T}$ . Then  $I$  is a two-sided ideal, and the algebra  $\mathbb{k}[\mathfrak{G}]/I$  is simple. In particular, if  $\mathfrak{G}$  is Hausdorff, then  $\mathbb{k}[\mathfrak{G}]$  is simple.*

*Proof.* The fact that  $I$  is a two-sided ideal follows directly from the fact that  $\mathcal{T}$  is  $\mathfrak{G}$ -invariant.

In order to prove simplicity of  $\mathbb{k}[\mathfrak{G}]$  it is enough to show that if  $f \in \mathbb{k}[\mathfrak{G}] \setminus I$ , then there exist elements  $a_i, b_i \in \mathbb{k}[\mathfrak{G}]$  such that  $\sum_{i=1}^k a_i f b_i = 1$ .

If  $f \in \mathbb{k}[\mathfrak{G}] \setminus I$ , then there exists  $g \in \mathfrak{G}$  such that  $\mathfrak{o}(g), \mathfrak{t}(g) \in \mathcal{T}$  and  $f(g) \neq 0$ . Let  $f = \sum_{i=1}^m \alpha_i 1_{F_i}$ , where  $F_i$  are open compact  $\mathfrak{G}$ -bisections. Let  $A = \{1 \leq i \leq m : g \in F_i\}$ . Then  $f(g) = \sum_{i \in A} \alpha_i$ . Since  $\mathfrak{o}(g) \in \mathcal{T}$ , an equality of targets  $\mathfrak{t}(F_i \mathfrak{o}(g)) = \mathfrak{t}(F_j \mathfrak{o}(g))$  implies the equality  $F_i \mathfrak{o}(g) = F_j \mathfrak{o}(g)$  of groupoid elements. It follows that  $\mathfrak{t}(F_i \mathfrak{o}(g)) \neq \mathfrak{t}(g)$  for every  $i \notin A$ . We can find therefore a clopen neighborhood  $U$  of  $\mathfrak{o}(g)$  such that  $U \subset \mathfrak{o}(F_i)$ ,  $F_i U = F_j U$ , for all  $i, j \in A$ ,  $U \cap \mathfrak{o}(F_j) = \emptyset$  for all  $j \notin A$ , and  $\mathfrak{t}(F_i U) \cap \mathfrak{t}(F_j U) = \emptyset$  for all  $i \in A$  and  $j \notin A$ . Denote  $F_i U = F$  for any  $i \in A$ . We have  $1_{F^{-1}} f 1_U = \sum_{i \in A} \alpha_i 1_U$ . It follows that  $1_U = \alpha 1_{F^{-1}} f 1_U$  for some  $\alpha \in \mathbb{k}$ .

The groupoid  $\mathfrak{G}$  is minimal, hence for every  $x \in \mathfrak{G}^{(0)}$  there exists  $h \in \mathfrak{G}$  such that  $\mathfrak{o}(h) = x$  and  $\mathfrak{t}(h) \in U$ . There exists therefore an open compact  $\mathfrak{G}$ -bisection  $H$  such that  $x \in \mathfrak{o}(H)$  and  $\mathfrak{t}(H) \subset U$ . Then  $1_{\mathfrak{o}(H)} = 1_{H^{-1}} 1_U 1_H = \alpha 1_{H^{-1} F^{-1}} f 1_U 1_H$ . It follows that  $\mathfrak{G}^{(0)}$  can be covered by a finite collection of sets  $V_i$  such that  $1_{V_i}$

can be written in the form  $a_i f b_i$  for some  $a, b \in \mathbb{k}[G]$ . Note that if  $V'_i$  is a clopen subset of  $V_i$ , then  $1_{V'_i} = 1_{V'_i} 1_{V_i}$ , hence we may replace the covering  $\{V_i\}$  by a finite covering by disjoint clopen sets. But in that case we have  $1 = \sum 1_{V_i}$ .  $\square$

#### 4.2. Growth of $\mathbb{k}[\mathfrak{G}]$ .

**Theorem 4.2.** *Let  $\mathfrak{G}$  be an étale groupoid with compact totally disconnected unit space. Let  $\mathcal{S}$  be a finite set of open compact  $\mathfrak{G}$ -bisections. Let  $V \subset \mathbb{k}[\mathfrak{G}]$  be the  $\mathbb{k}$ -subspace generated by the characteristic functions of the elements of  $\mathcal{S}$ . Then*

$$\dim V^n \leq \overline{\gamma}(n, \mathcal{S}) \delta(n, \mathcal{S}).$$

*Proof.* Fix  $n$ , and let  $\mathcal{S}^n$  be the set of all products  $S_1 S_2 \dots S_n$  of length  $n$  of elements of  $\mathcal{S}$ . Then  $V^n$  is the linear span of the characteristic functions of elements of  $\mathcal{S}^n$ . Denote, for  $x \in \mathfrak{G}^{(0)}$ ,

$$A_x = \bigcap_{F \in \mathcal{S}^n, x \in \mathfrak{o}(F)} \mathfrak{o}(F) \setminus \bigcup_{F \in \mathcal{S}^n, x \notin \mathfrak{o}(F)} \mathfrak{o}(F).$$

Since  $\mathfrak{o}(F)$  is clopen for every  $F \in \mathcal{S}^n$ , the sets  $A_x$  are also clopen. Note that for every  $F \in \mathcal{S}^n$  and  $x \in \mathfrak{G}^{(0)}$ , either  $A_x \subset \mathfrak{o}(F)$ , or  $A_x \cap \mathfrak{o}(F) = \emptyset$ .

If  $F_1, F_2$  are open  $\mathfrak{G}$ -bisections and  $F_1 \cdot x = F_2 \cdot x$  for a unit  $x$ , then the set of points  $y$  such that  $F_1 \cdot y = F_2 \cdot y$  is equal to the intersection of  $F_1^{-1} F_2$  with  $\mathfrak{G}^{(0)}$ . Since  $\mathfrak{G}$  is étale, this set is open. Denote by  $B_x$  the set of all points  $y \in A_x$  such that  $F_1 \cdot x = F_2 \cdot x$  implies  $F_1 \cdot y = F_2 \cdot y$  for all  $F_1, F_2 \in \mathcal{S}^n$ . Then  $B_x$  is open and  $x \in B_x$ .

Note that if  $x \sim_n y$ , then  $A_x = A_y$ , as belonging of a point  $y$  to the domain of a product  $S_1 S_2 \dots S_n$  of elements of  $\mathcal{S}$  is equivalent to the existence of a path in  $\mathfrak{G}(y, \mathcal{S})$  of length  $n$  starting at  $y$  and labeled by the sequence  $S_n, S_{n-1}, \dots, S_1$ . Similarly, if  $x \sim_n y$ , then  $B_x = B_y$ , since an equality  $F_1 \cdot x = F_2 \cdot x$  is equivalent to coincidence of endpoints of the paths corresponding to the products  $F_1$  and  $F_2$  starting at  $x$ .

Let  $\mathcal{B} = \{B_x : x \in \mathfrak{G}^{(0)}\}$ . Since  $B_x = B_y$  for  $x \sim_n y$ , the set  $\mathcal{B}$  consists of at most  $\delta(n, \mathcal{S})$  elements.

**Lemma 4.3.** *There exists a covering  $\tilde{\mathcal{B}} = \{\tilde{B}\}_{B \in \mathcal{B}}$  of  $\mathfrak{G}^{(0)}$  by disjoint clopen sets such that  $\tilde{B} \subset B$  for every  $B \in \mathcal{B}$ .*

We allow some of the sets  $\tilde{B}$  to be empty.

*Proof.* By the Shrinking Lemma, we can find for every  $B \in \mathcal{B}$  an open set  $B' \subset B$  such that  $\{B'\}_{B \in \mathcal{B}}$  is a covering of  $\mathfrak{G}^{(0)}$ , and closure of  $B'$  is contained in  $B$ . Then closure of  $B'$  is compact, and can be covered by a finite collection of clopen subsets of  $B$ . Hence, after replacing  $B'$  by the union of these clopen subsets, we may assume that  $B'$  are clopen. Order the set  $\mathcal{B}$  into a sequence  $B_1, B_2, \dots, B_m$ , define  $\tilde{B}_1 = B'_1$ , and inductively,  $\tilde{B}_i = B'_i \setminus (B'_1 \cup B'_2 \cup \dots \cup B'_{i-1})$ . Then  $\{\tilde{B}\}_{B \in \mathcal{B}}$  satisfies the conditions of the lemma.  $\square$

Let  $x_1, x_2, \dots, x_m$  be a transversal of the  $\sim_n$  equivalence relation, where  $m = \delta(n, \mathcal{S})$ . For every  $F \in \mathcal{S}^n$  and  $x_i \in \mathfrak{o}(F)$ , consider the restriction  $F \cdot \tilde{B}_{x_i}$  of  $F$  onto  $\tilde{B}_{x_i}$ . Since  $\{\tilde{B}_{x_i}\}_{i=1, \dots, m}$  is a covering of  $\mathfrak{G}^{(0)}$  by disjoint subsets, the sets  $F \cdot \tilde{B}_{x_i}$  form a covering of  $F$  by disjoint subsets, and  $1_F = \sum_{i=1}^m 1_{F \cdot \tilde{B}_{x_i}}$ .

If  $F_1, F_2 \in \mathcal{S}^n$  and  $x_i$  are such that  $x_i \in \mathfrak{o}(F_1) \cap \mathfrak{o}(F_2)$ , and  $F_1 \cdot x_i = F_2 \cdot x_i$ , then for every  $y \in \tilde{B}_{x_i}$  we have  $y \in \mathfrak{o}(F_1) \cap \mathfrak{o}(F_2)$  and  $F_1 \cdot y = F_2 \cdot y$ , hence  $F_1 \cdot \tilde{B}_{x_i} = F_2 \cdot \tilde{B}_{x_i}$ . It follows that  $F \cdot \tilde{B}_{x_i}$  depends only on  $F \cdot x_i$ , and we have not more than  $\gamma(n, x_i, \mathcal{S}) \leq \bar{\gamma}(n, \mathcal{S})$  non-empty sets of the form  $F \cdot \tilde{B}_{x_i}$ , for every given  $x_i$ . Hence we have at most  $\bar{\gamma}(n, \mathcal{S})\delta(n, \mathcal{S})$  functions of the form  $1_{F \cdot x_i}$  in total, and every function  $1_F$ , for  $F \in \mathcal{S}^n$  is equal to the sum of a subset of these functions, which finishes the proof of the theorem.  $\square$

**4.3. Finite generation.** For a given finite set  $\mathcal{S}$  of open compact  $\mathfrak{G}$ -bisections, generating  $\mathfrak{G}$ , denote

$$A_{x,n} = \bigcap_{F \in \mathcal{S}^n, x \in \mathfrak{o}(F)} \mathfrak{o}(F) \setminus \bigcup_{F \in \mathcal{S}^n, x \notin \mathfrak{o}(F)} \mathfrak{o}(F),$$

see the proof of Theorem 4.2. Recall that the sets  $A_{x,n}$  are clopen. It is also easy to see that two sets  $A_{x,n}$  and  $A_{y,n}$  are either disjoint or coincide. Note also that  $A_{x,n} \subset A_{x,m}$  if  $n > m$ . It follows that for any  $x, y \in \mathfrak{G}^{(0)}$  and  $n > m$ , either  $A_{x,n} \subset A_{y,m}$ , or  $A_{x,n} \cap A_{y,m} = \emptyset$ .

**Definition 4.3.** We say that  $\mathcal{S}$  is *expansive* if for any two different points  $x, y \in \mathfrak{G}^{(0)}$  there exists  $n$  such that  $A_{x,n}$  and  $A_{y,n}$  are disjoint.

**Proposition 4.4.** *If  $\mathcal{S}$  is expansive, then the set  $\{1_S : S \in \mathcal{S} \cup \mathcal{S}^{-1}\}$  generates  $\mathbb{k}[\mathfrak{G}]$ .*

*Proof.* Let  $\mathcal{A}$  be the algebra generated by the functions  $1_S$  for  $S \in \mathcal{S} \cup \mathcal{S}^{-1}$ . Note that  $\mathfrak{o}(F) = F^{-1}F$ , hence  $1_F \in \mathcal{A}$  for every  $F \in (\mathcal{S} \cup \mathcal{S}^{-1})^n$ . Note also that  $1_{A \cap B} = 1_A \cdot 1_B$ ,  $1_{A \setminus B} = 1_A \cdot (1_A - 1_B)$ , and  $1_{A \cup B} = 1_A + 1_B - 1_A 1_B$  for every  $A, B \subset \mathfrak{G}^{(0)}$ . It follows that  $1_{A_{x,n}} \in \mathcal{A}$  for all  $x \in \mathfrak{G}^{(0)}$  and  $n$ .

Let us show that for every open set  $A \subset \mathfrak{G}^{(0)}$  and every  $x \in A$  there exists  $n$  such that  $A_{x,n} \subset A$ . For every  $y \notin A$  there exists  $n_y$  such that  $A_{x,n_y} \cap A_{y,n_y} = \emptyset$ . Since  $\mathfrak{G}^{(0)} \setminus A$  is compact, there exists a finite covering  $A_{y_1, n_{y_1}}, A_{y_2, n_{y_2}}, \dots, A_{y_m, n_{y_m}}$  of  $\mathfrak{G}^{(0)} \setminus A$ . Let  $n = \max n_{y_i}$ . Then  $A_{x,n} \subset A$ .

Let  $F$  be an arbitrary open compact  $\mathfrak{G}$ -bisection. For every  $g \in F$  there exists  $n$  and  $F' \in (\mathcal{S} \cup \mathcal{S}^{-1})^n$  such that  $g \in F'$ . There also exists  $n_g$  such that  $A_{\mathfrak{o}(g), n_g} \subset \mathfrak{o}(F)$  and  $F \cdot A_{\mathfrak{o}(g), n_g} = F' \cdot A_{\mathfrak{o}(g), n_g}$ . We get a covering of  $F$  by sets of the form  $F' \cdot A_{x,m}$ , where  $F' \in (\mathcal{S} \cup \mathcal{S}^{-1})^n$ . Since any two sets of the form  $A_{x,n}$  are either disjoint or one is a subset of the other, we can find a covering of  $F$  by disjoint sets of the form  $F' \cdot A_{x,m}$  for  $F' \in (\mathcal{S} \cup \mathcal{S}^{-1})^n$ . This implies that  $1_F \in \mathcal{A}$ , which finishes the proof.  $\square$

#### 4.4. Examples.

**4.4.1. Subshifts.** Let  $\mathcal{X} \subset X^{\mathbb{Z}}$  be a subshift, and let  $\mathfrak{S}$  be the groupoid of germs generated by the shift  $s : \mathcal{X} \rightarrow \mathcal{X}$ . Let, as in 3.3.1,  $S_x = \{(s, w) : w(0) = x\}$ ,  $\mathcal{S} = \{S_x\}_{x \in X}$ . Note that for every word  $x_1 x_2 \dots x_n$  domain of the product  $S_{x_1} S_{x_2} \dots S_{x_n}$  is the set of words  $w \in \mathcal{X}$  such that  $w(0) = x_n, w(1) = x_{n-1}, \dots, w(n-1) = x_1$ . It follows that the set  $\mathcal{S} \cup \mathcal{S}^{-1}$  is expansive, and by Proposition 4.4,  $\{1_S\}_{S \in \mathcal{S} \cup \mathcal{S}^{-1}}$  is a generating set of  $\mathbb{k}[\mathfrak{S}]$ .

Since  $\mathfrak{S}$  coincides with the groupoid of the  $\mathbb{Z}$ -action on  $\mathcal{X}$  defined by the shift, the algebra  $\mathbb{k}[\mathfrak{S}]$  is the corresponding cross-product of the algebra of continuous  $\mathbb{k}$ -valued functions with the group algebra of  $\mathbb{Z}$ . Every its element is uniquely written

as a Laurent polynomial  $\sum a_n \cdot t^n$ , where  $t \in \mathbb{k}[\mathfrak{G}]$  is the characteristic function of the set of germs of the shift  $s : \mathcal{X} \rightarrow \mathcal{X}$ , and  $a_n$  are continuous  $\mathbb{k}$ -valued functions. Multiplication rule for such polynomials follows from the relations  $t \cdot a = b \cdot t$ , where  $a, b : \mathcal{X} \rightarrow \mathbb{k}$  satisfy  $b(w) = a(s^{-1}(w))$  for every  $w \in \mathcal{X}$ .

**Proposition 4.5.** *Let  $V$  be the linear span of  $\{1\} \cup \{1_S\}_{S \in \mathcal{S} \cup \mathcal{S}^{-1}}$ . Then*

$$\left\lfloor \frac{n}{2} \right\rfloor p_{\mathcal{X}} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \leq \dim V^n \leq (2n+1)p_{\mathcal{X}}(2n).$$

*Proof.* The upper bound follows from Theorem 4.2. For the lower bound note that  $S_{x_1}S_{x_2}\dots S_{x_n}$  and  $S_{y_1}S_{y_2}\dots S_{y_m}$  are disjoint if  $x_1x_2\dots x_n \neq y_1y_2\dots y_m$ , hence the set of characteristic functions of all non-zero products of elements of  $\mathcal{S}$  is linearly independent, so that  $\sum_{k=0}^n p_{\mathcal{X}}(k) \leq \dim V^n$ . Since  $p_{\mathcal{X}}(n)$  is non-decreasing, we have  $\left\lfloor \frac{n}{2} \right\rfloor p_{\mathcal{X}} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \leq \sum_{k=0}^n p_{\mathcal{X}}(k)$ .  $\square$

Note that since the characteristic functions of the products  $S_{x_1}S_{x_2}\dots S_{x_n}$  are linearly independent, their linear span is a sub-algebra of  $\mathbb{k}[\mathfrak{G}]$  isomorphic to the semigroup algebra  $\mathcal{M}_{\mathcal{X}}$  of the semigroup generated by the set  $\{S_x : x \in X\}$ . It is easy to see that  $\mathcal{M}_{\mathcal{X}}$  is isomorphic to the quotient of the free associative algebra generated by  $X$  modulo the ideal generated by all words  $w \in X^*$  such that  $w$  is not a subword of any element of the subshift  $\mathcal{X}$ . It follows from Proposition 4.5 that growths of  $\mathbb{k}[\mathfrak{G}]$  and  $\mathcal{M}_{\mathcal{X}}$  are equivalent. Note that the algebras  $\mathcal{M}_{\mathcal{X}}$  are the original examples of algebras of arbitrary Gelfand-Kirillov dimension, see [34] and [19, Theorem 2.9].

**Example 4.2.** Let  $\mathcal{X}$  be a Sturmian subshift. It is minimal and  $p_{\mathcal{X}}(n) = n + 1$ , hence

$$\frac{(n+1)(n+2)}{2} \leq \dim V^n \leq 2n(2n+1),$$

so that  $\mathbb{k}[\mathfrak{G}]$  is a quadratically growing finitely generated algebra. Note that it is simple by Proposition 4.1. This disproves Conjecture 3.1 in [4].

**Example 4.3.** It is easy to see that every Toeplitz subshift is minimal. Consequently, known examples of Toeplitz subshifts (see Subsection 3.3.1) provide us with simple finitely generated algebras of arbitrary Gelfand-Kirillov dimension  $\alpha \geq 2$ , and also uncountably many different growth types of simple finitely generated algebras of Gelfand-Kirillov dimension two (see a question on existence of such algebras on page 832 of [5]).

**4.4.2. Self-similar groups.** Let  $G$  be a self-similar group of automorphisms of the tree  $X^*$ . Let  $\mathfrak{G}$  be the groupoid of germs of its action on the boundary  $X^{\mathbb{N}}$  of the tree. Suppose that  $G$  is *self-replicating*, i.e., for all  $x, y \in X$  and  $g \in G$  there exists  $h \in G$  such that  $g(x) = y$  and  $h|_x = g$ . Then for all pairs of words  $v, u \in X^*$  of equal length and every  $g \in G$  there exists  $h \in G$  such that  $h(v) = u$  and  $h|_v = g$ . In other words, the transformation  $vw \mapsto ug(w)$  is an open compact  $\mathfrak{G}$ -bisection (more pedantically, the set of its germs is a bisection, but we will identify a  $\mathfrak{G}$  bisection  $F$  with the map  $\mathfrak{o}(g) \mapsto \mathfrak{t}(g)$ ,  $g \in F$ ).

Fix  $n \geq 0$ , and consider the set of all  $\mathfrak{G}$ -bisections of the form  $R_{u,g,v} : vw \mapsto ug(w)$  for  $v, u \in X^n$  and  $g \in G$ . Note that these bisections are multiplied by the rule

$$(1) \quad R_{u_1,g_1,v_1} R_{u_2,g_2,v_2} = \begin{cases} 0 & \text{if } v_1 \neq u_2; \\ R_{u_1,g_1g_2,v_2} & \text{if } v_1 = u_2. \end{cases}$$

Let  $A_n$  be the formal linear span of the elements  $R_{u,g,v}$  for  $u, v \in X^n$  and  $g \in G$ . Extend multiplication rule (1) to  $A_n$ . It is easy to see then that  $A_n$  is isomorphic to the algebra  $M_{d^n \times d^n}(\mathbb{k}[G])$  of matrices of size  $d^n \times d^n$  over the group ring  $\mathbb{k}[G]$ .

The map  $R_{u,g,v} \mapsto \sum_{x \in X} R_{ug(x),g|x,vx}$  induces a homomorphism  $A_n \mapsto A_{n+1}$  called the *matrix recursion*. More on matrix recursions for self-similar groups see [3, 2, 23, 25, 11].

**Example 4.4.** For the adding machine action (see Example 3.3) the matrix recursions replace every entry  $a^n$  by  $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}^n$ , i.e., are induced by the map

$$a \mapsto \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}.$$

For example, the image of  $a$  in  $A_2$  is

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

For the Grigorchuk group the matrix recursions are induced by the map

$$\begin{aligned} a &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & b &\mapsto \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \\ c &\mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, & d &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}. \end{aligned}$$

**Proposition 4.6.** *The convolution algebra  $\mathbb{k}[\mathfrak{G}]$  of the groupoid of germs of the action of  $G$  on  $X^{\mathbb{N}}$  is isomorphic to the direct limit of the matrix algebras  $A_n \cong M_{d^n \times d^n}(\mathbb{k}[G])$  with respect to the matrix recursions.*

*Proof.* Denote by  $A_\infty$  the direct limit of the algebras  $A_n$  with respect to the matrix recursions. Let  $\phi : A_\infty \rightarrow \mathbb{k}[\mathfrak{G}]$  be the natural map given by  $\phi(R_{u,g,v}) = 1_{R_{u,g,v}}$ . Note that  $1_{R_{u,g,v}} = \sum_{x \in X} 1_{R_{ug(x),g|x,vx}}$ , hence the map  $\phi$  is well defined. It also follows from equation (1) that  $\phi$  is a homomorphism of algebras. It remains to show that  $\phi$  is injective. Let  $f$  be a non-zero element of  $\mathbb{k}[\mathfrak{G}]$ , and let  $(g, w) \in \mathfrak{G}$  be such that  $f(g, w) \neq 0$ . Suppose that  $\phi(f) = \sum_{u,v \in X^n} \alpha_{u,v} R_{u,g_{u,v},v}$  for some  $\alpha_{u,v} \in \mathbb{k}$  and  $g_{u,v} \in G$ . Denote the set of all pairs  $(u, v)$  such that  $(g, w) \in R_{u,g_{u,v},v}$  and  $\alpha_{u,v} \neq 0$  by  $P$ . The set  $\bigcap_{(u,v) \in P} R_{u,g_{u,v},v}$  is an open neighborhood of  $(g, w)$ , hence there exists a  $\mathfrak{G}$ -bisection  $R_{w_1,h,w_2}$  contained in  $\bigcap_{(u,v) \in P} R_{u,g_{u,v},v}$ . Applying the matrix recursion, we get a representation of  $f$  as an element  $\sum_{u,v \in X^{|w_1|}} \beta_{u,v} R_{u,h_{u,v},v} \in A_{|w_1|}$  such that  $(g, w)$  does not belong to any set  $R_{u,h_{u,v},v}$ ,  $u, v \in X^{|w_1|}$ ,  $(u, v) \neq (w_1, w_2)$ . Then  $f(g, w) = \beta_{u,v} \neq 0$ , hence  $\phi(f) \neq 0$ .  $\square$

As a corollary of Proposition 3.3 and Theorem 4.2 we get the following result of L. Bartholdi [2].

**Proposition 4.7.** *Let  $G$  be a contracting self-replicating group, and let  $\mathfrak{G}$  be the groupoid of germs of its action on  $X^{\mathbb{N}}$ . Every finitely generated sub-algebra of  $\mathbb{k}[\mathfrak{G}]$  has Gelfand-Kirillov dimension at most  $\frac{2 \log |X|}{-\log \lambda}$ , where  $\lambda$  is the contraction coefficient of  $G$ .*



The image of the group ring  $\mathbb{k}[G]$  in  $\mathbb{k}[\mathfrak{G}]$  is called the *thinned algebra*. It was defined in [31], see also [2].

Let us come back to the case of the Grigorchuk group. Since its contraction coefficient is equal to  $1/2$ , every finitely generated sub-algebra of  $\mathbb{k}[\mathfrak{G}]$  has Gelfand-Kirillov dimension at most 2. It is easy to prove that it is actually equal to 2 in this case. Moreover, it has quadratic growth, see [2].

This example is also an illustration of the non-Hausdorffness phenomenon. The groupoid of germs of the Grigorchuk group is not Hausdorff: the germs  $(b, 111\dots)$ ,  $(c, 111\dots)$ ,  $(d, 111\dots)$ , and  $(1, 111\dots)$  do not have disjoint neighborhoods.

**Example 4.5.** Consider the convolution algebra  $\mathbb{F}_2[\mathfrak{G}]$  for the groupoid of germs of the Grigorchuk group over the field with two elements. The matrix recursion for the element  $b + c + d + 1$  is

$$b + c + d + 1 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & b + c + d \end{pmatrix}.$$

It follows that  $b + c + d$  is a non-trivial element of  $\mathbb{F}_2[\mathfrak{G}]$  but, as a function on  $\mathfrak{G}$  is zero everywhere except for the germs of  $b, c, d, 1$  at  $111\dots$ , where it is equal to 1. This shows that the ideal  $I$  from Proposition 4.1 is non-zero in this case, and the algebra  $\mathbb{F}_2[\mathfrak{G}]$  is not simple.

**4.5. Modules  $\mathbb{k}\mathfrak{G}_x$ .** Let  $\mathfrak{G}$  be an étale minimal groupoid. Consider the space  $\mathbb{k}\mathfrak{G}_x$  of maps  $\phi : \mathfrak{G}_x \rightarrow \mathbb{k}$  with finite support, where  $\mathfrak{G}_x = \{g \in \mathfrak{G} : o(g) = x\}$ . It is easy to see that for every  $\phi \in \mathbb{k}\mathfrak{G}_x$  and  $f \in \mathbb{k}[\mathfrak{G}]$  the convolution  $f \cdot \phi$  is an element of  $\mathbb{k}\mathfrak{G}_x$ , and that  $\mathbb{k}\mathfrak{G}_x$  is a left  $\mathbb{k}[\mathfrak{G}]$ -module with respect to the convolution.

**Proposition 4.8.** *Let  $\mathcal{S}$  be a finite set of open compact  $\mathfrak{G}$ -bisections, and let  $V \subset \mathbb{k}[\mathfrak{G}]$  be the linear span of their characteristic functions and  $1_{\mathfrak{G}(o)}$ . Then for every  $n \geq 1$  we have*

$$\dim V^n \cdot \delta_x \leq \gamma_{\mathcal{S}}(x, n),$$

where  $\delta_x \in \mathbb{k}\mathfrak{G}_x$  is the characteristic function of  $x \in \mathfrak{G}_x$ , and  $\gamma_{\mathcal{S}}(x, n)$  is the growth of the Cayley graph based at  $x$  of the groupoid generated by the union of the elements of  $\mathcal{S}$ .

*If the isotropy group of  $x$  is trivial, then the module  $\mathbb{k}\mathfrak{G}_x$  is simple.*

*Proof.* The growth estimate is obvious, since for every  $g \in \mathfrak{G}_x$  and  $S \in \mathcal{S}$  we have  $1_S \cdot \delta_g = \delta_{Sg}$ , if  $Sg \neq \emptyset$ , and  $1_S \cdot \delta_g = 0$  otherwise.

Let us show that  $\mathbb{k}\mathfrak{G}_x$  is simple if the isotropy group of  $x$  is trivial. It is enough to show that for every non-zero element  $\phi \in \mathbb{k}\mathfrak{G}_x$  there exist elements  $f_1, f_2 \in \mathbb{k}[\mathfrak{G}]$  such that  $f_1 \cdot \phi = \delta_x$  and  $f_2 \cdot \delta_x = \phi$ .

Let  $\phi \in \mathbb{k}\mathfrak{G}_x$ , and let  $\{g_1, g_2, \dots, g_k\}$  be the support of  $\phi$ . Since the isotropy group of  $x$  is trivial,  $t(g_i)$  are pairwise different. Let  $U_1, U_2, \dots, U_k$  be open compact  $\mathfrak{G}$ -bisections such that  $g_i \in U_i$  and  $t(U_i)$  are disjoint. Then  $\left(\sum_{i=1}^k \phi(g_i) 1_{U_i}\right) \cdot \delta_x = \phi$  and  $\phi(g_1)^{-1} 1_{U_1^{-1}} \phi = \delta_x$ .  $\square$

**Example 4.6.** Let  $X$  be a finite alphabet, and let  $w \in X^{\mathbb{Z}}$  be a non-periodic sequence such that closure  $\mathcal{X}_w$  of the shift orbit of  $w$  is minimal. Let  $\mathfrak{G}$  be the groupoid generated by the action of the shift on  $\mathcal{X}_w$ . Denote by  $T$  and  $T^{-1}$  the characteristic functions of the sets of germs of the shift and its inverse, and for every  $x \in X$ , denote by  $D_x$  the characteristic function of the cylindrical set  $\{w \in \mathcal{X}_w : w(0) = x\}$ . Then  $\mathbb{k}[\mathfrak{G}]$  is generated by  $T, T^{-1}$  and  $D_x$  for  $x \in X$ . Note that

we can remove one of the generators  $D_x$ , since  $\sum_{x \in X} D_x = 1 = TT^{-1}$ . Consider the set  $\mathfrak{S}_w = \{(s^n, w) : n \in \mathbb{Z}\}$  and the corresponding module  $\mathbb{k}\mathfrak{S}_w$ . Its basis as a  $\mathbb{k}$ -vector space consists of the delta-functions  $e_n = \delta_{(s^n, w)}$ ,  $n \in \mathbb{Z}$ . In this naturally ordered basis left multiplication by  $T$  is given by the matrix

$$T = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (t_{ij})_{i \in \mathbb{Z}, j \in \mathbb{Z}}$$

with the entries  $t_{m,n} = \delta_{m-1,n}$ . The element  $T^{-1}$  is given by the transposed matrix, and an element  $D_x$  is given by the diagonal matrix  $(a_{ij})$  with entries given by the rule

$$a_{nn} = \begin{cases} 1 & \text{if } w(n) = x, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the algebra  $\mathbb{k}[\mathfrak{S}]$  is isomorphic to the algebra generated by such matrices. For example, if  $X = \{0, 1\}$ , then the algebra is generated by the matrices  $T$ ,  $T^\top$ , and the diagonal matrix with the sequence  $w$  on the diagonal.

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